Chapter 3 Vector Space

(Scalar) multiplication and (vector) addition in $\mathbb{R}^n$ are defined by

$$\alpha\mathbf{x} = \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix} \quad \text{and} \quad \mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}$$

for any $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^n$ and any scalar α.

Definition: Let V be a set with elements defined over a field F (usually, $\mathbb{R}$ or $\mathbb{C}$). Let "+" and "•" be the addition and scalar multiplication operations defined on V, respectively. Then $(V, +, \bullet)$ forms a vector space if axioms A1 ~ A8 hold.

A1. $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$ for any $\mathbf{x}$ and $\mathbf{y}$ in V.

A2. $(\mathbf{x}+\mathbf{y})+\mathbf{z}=\mathbf{x}+(\mathbf{y}+\mathbf{z})$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in V.

A3. There exists an element $\mathbf{0}$ (or denoted by $\mathbf{0}_V$) in V such that $\mathbf{x}+\mathbf{0}=\mathbf{x}$ for any $\mathbf{x}\in V$.

A4. For any $\mathbf{x}\in V$, there exists an element $\text{-}\mathbf{x}$ in V such that $\mathbf{x}+(\text{-}\mathbf{x})=\mathbf{0}$. (By A1, it can be written as $(\text{-}\mathbf{x})+\mathbf{x}=\mathbf{0}$ as well.)

A5. $\alpha\bullet(\mathbf{x}+\mathbf{y})=\alpha\bullet\mathbf{x}+\alpha\bullet\mathbf{y}$ for each real number α and any $\mathbf{x}$ and $\mathbf{y}$ in V.

A6. $(\alpha+\beta)\bullet\mathbf{x}=\alpha\bullet\mathbf{x}+\beta\bullet\mathbf{x}$ for any real numbers α and β and any $\mathbf{x}\in V$.

A7. $(\alpha\beta)\bullet\mathbf{x}=\alpha\bullet(\beta\bullet\mathbf{x})$ for any real numbers α and β and any $\mathbf{x}\in V$.

A8. $1\bullet\mathbf{x}=\mathbf{x}$ for all $\mathbf{x}\in V$.

Note: *Two important closure properties* implicitly existed in these axioms are:

C1. If $\mathbf{x} \in V$ and α is a scalar, then $\alpha\mathbf{x} \in V$ (with $\alpha\mathbf{x} := \alpha \bullet \mathbf{x}$).

C2. If $\mathbf{x}, \mathbf{y} \in V$, then $\mathbf{x} + \mathbf{y} \in V$.

(*Because, without C1 and C2, the scalar products in A5 ~ A7 and the vector addition in A1 all become meaningless, respectively*.) Use the set

$$W = \{(a, 1) \mid a \in \mathbb{R}\}$$

as an example to explain the failure of the closure properties. And explain why A1 ~ A8 can't all be true for W with operations + and $\bullet$. So, $(W, +, \bullet)$ is not a vector space.

- We call elements in V vectors. However, vectors need not be *real* vectors in $\mathbb{R}^n$.
  ($\mathbb{R}^n$ 被稱之為歐基里德向量空間, Euclidean vector space)

- $\mathbb{R}^{m\times n}$ with the usual matrix addition and scalar multiplication is a vector space

- Denote $C[a,b]$ the set of all real-valued functions that are defined and *continuous* on the closed interval $[a,b]$. Then $(C[a,b],+,\bullet)$ forms a vector space with the usual definitions of "+" and "•" as

$$\begin{aligned}(f+g)(x) &= f(x)+g(x)\\ (\alpha f)(x) &= \alpha f(x)\end{aligned}$$

Now vectors are *continuous functions* in $C[a,b]$.

- Denote P_n the set of all polynomials of degree less than n. Define

$$\begin{aligned}(p+q)(x) &= p(x)+q(x)\\ (\alpha p)(x) &= \alpha p(x)\end{aligned}$$

$(P_n,+,\bullet)$ is also a vector space with vectors being *polynomials*.

Theorem 3.1.1: If V is a vector space and $\mathbf{x} \in V$ is arbitrary, then

(i) $0\mathbf{x} = \mathbf{0}_V$

(ii) $\mathbf{x} + \mathbf{y} = \mathbf{0}_V$ implies that $\mathbf{y} = \mathbf{-x}$

(iii) $(-1)\mathbf{x} = \mathbf{-x}$

Explanation: (i) says that multiplying the scalar 0 *to any* $\mathbf{x} \in V$ *gets the zero vector* $\mathbf{0}_V$ *of* V.

(ii) says that if $\mathbf{x} + \mathbf{y} = \mathbf{0}_V$, *then* $\mathbf{y}$ *is the additive inverse of* $\mathbf{x}$.

(iii) says that the additive inverse of $\mathbf{x}$ *can be obtained by multiplying* $\mathbf{x}$ *with the scalar* (-1).

Exercises 7 & 8: (in p. 122) Let V be a vector space with arbitrary vectors $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$. Show that

(1) The element $\mathbf{0}$ in V is unique. (加法單位向量的唯一性)

(2) If $\mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z}$, then $\mathbf{y} = \mathbf{z}$. (消除法可適用）

Proof: For (1): Let $\boldsymbol{\delta} \neq \mathbf{0}$ *be another unit of addition satisfying A3, i.e. for* $\forall\, \mathbf{x} \in V$, $\mathbf{x} + \boldsymbol{\delta} = \mathbf{x}$. *Set* $\mathbf{x} = \mathbf{0}$, *by A1 we get a contradiction.*

For (2): By A4, $\exists\, \text{-}\mathbf{x} \in V$ *such that* $\mathbf{x} + (\text{-}\mathbf{x}) = \mathbf{0}$. *Thus*

$$\begin{aligned} \mathbf{x} + \mathbf{y} = \mathbf{x} + \mathbf{z} \;&\Rightarrow\; (\text{-}\mathbf{x}) + (\mathbf{x} + \mathbf{y}) = (\text{-}\mathbf{x}) + (\mathbf{x} + \mathbf{z}) \\ &\Rightarrow\; (\text{-}\mathbf{x} + \mathbf{x}) + \mathbf{y} = (\text{-}\mathbf{x} + \mathbf{x}) + \mathbf{z} \\ &\Rightarrow\; \mathbf{0} + \mathbf{y} = \mathbf{0} + \mathbf{z} \\ &\Rightarrow\; \mathbf{y} = \mathbf{z}. \end{aligned}$$

□

註：同學可嘗試練習做 **Exercise** 9 看看。

§ 3-2 Subspaces

Given a vector space V, it is often possible to form another vector space by taking a subset S of V and *using the operations of* V.

Example 1: Let $S = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \middle| \ x_2 = x_1 \right\}$, then $S \subset \mathbb{R}^2$. It is easy to see that

- any $\mathbf{x} \in S$ and any scalar $\alpha \Rightarrow \alpha\mathbf{x} \in S$
- any $\mathbf{x} \in S$ and any $\mathbf{y} \in S \Rightarrow \mathbf{x} + \mathbf{y} \in S$.

Moreover, A1 ~ A8 can be verified for $(S, +, \bullet)$. So, it is a vector space.

Definition: If S is a nonempty subset of a vector space V, and S satisfies the following conditions:

(i) $\alpha\mathbf{x} \in S$ whenever $\mathbf{x} \in S$ for any scalar α

(ii) $\mathbf{x} + \mathbf{y} \in S$ whenever $\mathbf{x} \in S$ and $\mathbf{y} \in S$

then S is said to be a subspace of V.

Two subsets $S = \{(x_1, x_2, x_3)^T \mid x_1 = x_2\}$ and $S = \{A \in \mathbb{R}^{2\times 2} \mid a_{12} = -a_{21}\}$ of $\mathbb{R}^3$ and $\mathbb{R}^{2\times 2}$ are given in Examples 2 and 4, respectively. As shown in the book, the two closure properties are true for both subsets. By the Definition, they are subspaces of $\mathbb{R}^3$ and $\mathbb{R}^{2\times 2}$, respectively. However, as shown in Example 3, though $S = \{(x, 1)^T \mid x \in \mathbb{R}\} \subset \mathbb{R}^2$, it is not a subspace of $\mathbb{R}^2$ because none of the two closure properties holds.

Example 6: Let $C^n[a,b]$ be the set of all functions that have a continuous nth derivative on $[a,b]$. Then, it is a subspace of $C[a,b]$.

Example 8: Let S be the set of all f in $C^2[a,b]$ such that

$$f''(x) + f'(x) = 0$$

for all $x \in [a,b]$. Since S contains at least the zero function thus is nonempty and it is easy to show that conditions (i) and (ii) in above Definition hold. So, S is a subspace of $C^2[a,b]$.

Exercise 5: (in p. 132) Determine whether or not the following are subspaces of P_4.

Soln.: For (a): The set S *of polys. in* P_4 *of even degree is not a subspace because, for* $f(x) := 2x^2 + x - 1$ *and* $g(x) := -2x^2 + x - 1$, *both are in* S, *but condition (ii) fails since* $f(x) + g(x) = 2x - 2 \notin S$.

For (d): The set S *of polys. in* P_4 *having at least one real root is not a subspace because, for* $f(x) := x^2 - 1$ *and* $g(x) := x + 2$, *both are in* S, *but condition (ii) fails due to* $f(x) + g(x) = x^2 + x + 1$, *which has roots* $(-1 \pm \sqrt{3}i)/2$, *thus* $f(x) + g(x) \notin S$. □

Exercise 8 (b): Let $A \in \mathbb{R}^{2\times 2}$ be given. Determine whether the set $S_2 := \{B \in \mathbb{R}^{2\times 2} \mid AB \neq BA\}$ is a subspace of $\mathbb{R}^{2\times 2}$.

Soln.: Note that $\mathbf{O} \notin S_2$. *However, this violates condition (i) because, for any* $B \in S_2$, $0 \cdot B = \mathbf{O}$. □

Exercise 20: (in p.134) Let U and V be subspaces of a vector space W. Define

$$U+V := \{\mathbf{z} \mid \mathbf{z} = \mathbf{u} + \mathbf{v} \quad \text{where } \mathbf{u} \in U, \text{ and } \mathbf{v} \in V\}.$$

Show that $U+V$ is a subspace of W.

Proof: Usually, nonemptiness of the subset S (i.e. $U+V$ here) is guaranteed. Let's prove the two closure conditions (i) and (ii) only.

For (i): For any $\mathbf{z} \in U+V$, $\mathbf{z} = \mathbf{u} + \mathbf{v}$ for some $\mathbf{u} \in U$ and $\mathbf{v} \in V$. Let α be an arbitrary scalar. Then

$$\alpha\mathbf{z} = \alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v} \in U+V$$

where "$\in$" is guaranteed by $\alpha\mathbf{u} \in U$ and $\alpha\mathbf{v} \in V$ and both U and V are vector spaces.

For (ii): For any $\mathbf{z}_1$ and $\mathbf{z}_2$ in $U+V$, we need to show $\mathbf{z}_1 + \mathbf{z}_2 \in U+V$. Can you practice to show it by yourselves? □

Definition: Let A be an $m \times n$ matrix. The **nullspace** of A, is the set of *all* solutions to the homogeneous system $A\mathbf{x} = \mathbf{0}$, i.e.

$$N(A) := \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

- $\{\mathbf{0}\} \subset N(A) \subset \mathbb{R}^n$ (集合間的包含關係)
- $N(A)$ is a vector space. Thus $N(A)$ is a subspace of $\mathbb{R}^n$. (書上 p.127 有證明)

Example 9: Let $A \in \mathbb{R}^{2\times 4}$ be given as in the book, determine $N(A)$.

Sol.: (1) Use elementary row operations to reduce A *into* A_{rref}. *(2) Set two free variables by* $x_3 = \alpha$, $x_4 = \beta$. *(3) Solve* x_1, x_2 *from equivalent system. (4) Use* α *and* β *to describe the solution to* $A\mathbf{x} = \mathbf{0}$ *as*

$$\mathbf{x} = \alpha \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + \beta \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Definition: Let $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ be vectors in a vector space V. The set of *all* linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$ is called the span of $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n$, denoted by $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$.

Theorem 3.2.1: $\mathrm{Span}(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ is a subspace of V.

Proof: See the proof of textbook in the class. The book doesn't prove Span$(\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n)$ *is a nonempty subset of* V. □

Definition: The set $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_n\}$ is a spanning set for V if, for *every* vector $\mathbf{v}$ in V, there exist scalars $c_1, c_2, \cdots, c_n$ such that $\mathbf{v} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_n\mathbf{v}_n$.

Example 11(a): The set $S = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, (1,2,3)^T\}$ spans $\mathbb{R}^3$. However, the vector $(1,2,3)^T$ is redundant (*since it is formed by a linear* *combination of the other 3 vectors*). Hence S is not a *minimal spanning set* for $\mathbb{R}^3$.

Example 12: The set $\{1-x^2,\ x+2,\ x^2\}$ spans P_3. (see textbook for the derivation)   **Q:** Is the set a minimal spanning set?