1. 97 台灣大學化學工程研究所

1. (15%) Under certain conditions it is found that the rate at which a solid substance dissolves varies directly as th product of the mount of undissolved solid present in the solvent and the difference between the saturation concentration and the instantaneous concentration of the substance. If 10 pounds of solute is dumped into a tank containing 100 pounds of solvent and at the end of 10 minutes the concentration is observed to be 1 part in 20, find the amount of solute in solution at any time t if the saturation concentration is 1 part of solute in 10 parts of solvent. 《97台大化工》

《解》 If Q is the amount of the material in solution at time t, then $(10-Q)$ is the amount of undissolved material present at that time and $\frac{Q}{100}$ is the corresponding concentration. 由題意可知

$$\frac{dQ}{dt}=k(10-Q)\left(\frac{1}{10}-\frac{Q}{100}\right) \tag{1}$$

且 $t=0$ 時 $Q=0$, 及 $t=10$ 時 $\frac{Q}{100}=\frac{1}{20}$ 即 $Q=5$, 現將 (1) 式分離變數可得

$$\frac{dQ}{(10-Q)^2}=\frac{k}{100} \tag{2}$$

對 (2) 式兩端積分可得

$$\frac{1}{10-Q}=\frac{k}{100}t+c \tag{3}$$

將 $t=0$ 時 $Q=0$ 代入 (3) 式, 可得 $c=\frac{1}{10}$, 再將 $t=10$ 時 $Q=5$ 代回 (3) 式, 可得 $k=1$, 故 (3) 式可改寫成

$$\frac{1}{10-Q}=\frac{t}{100}+\frac{1}{10}=\frac{t+10}{100}$$

即

$$10-Q=\frac{100}{t+10}\Rightarrow Q=10-\frac{100}{t+10}$$

或

$$Q = \frac{10t}{t+10}$$

2. A ball of mass m is thrown vertically downward from a building h feet high. The initial velocity of the ball is v_0. Suppose the air resistance can be neglected.
   (a) (10%) Show that the ball will impact on the ground at time $(\sqrt{v_0^2+2gh}-v_0)/g$.
   (b) (10%) Suppose ball 1 with $m_1 = 2$ pounds is dropped downward from the building with zero initial velocity. After it has fallen k feet $(k < h)$. ball 2 with $m_2 = 4$ pounded is dropped downward from the same point with zero initial velocity. Show that, when the first ball hits the ground, the second ball still has $(2\sqrt{hk}-k)$ feet to go. 《97台大化工》

《解》

(a) 因 $h = v_0 t + \frac{1}{2}gt^2$, 可解得

$$t = \frac{-v_0 \pm \sqrt{v_0^2+2gh}}{g} = \frac{-v_0 + \sqrt{v_0^2+2gh}}{g} \quad (\text{負不合})$$

(b) 第2顆球落下的時間 $t_2 =$ 第1顆球碰到地減去第1顆球落下 k 呎的時間, 故

$$t_2 = \sqrt{\frac{2h}{g}} - \sqrt{\frac{2k}{g}}$$

因此第1顆球碰到地時, 第2顆球距地面 h_1 feet, 則

$$h_1 = h - \frac{1}{2}g(t_2)^2 = h - \{\frac{g}{2}(\frac{2h}{g} + \frac{2k}{g} - \frac{2\sqrt{4hk}}{g}\} = 2\sqrt{hk} - k$$

3. (15%) Consider the initial value problem

$$\frac{d^2x(t)}{dt^2} + x(t) = f(t)\ ,\ t \geq 0\ ,\ x(0) = \frac{dx(0)}{dt} = 0$$

where $f(t) = \begin{cases} t & 0 < t \leq 1 \\ 1 & t > 1 \end{cases}$. Find the solution by means of Laplace transforms. 《97台大化工》

《解》 因

$$f(t) = \begin{cases} t & 0 < t \leq 1 \\ 1 & t > 1 \end{cases} = t - (t-1)u(t-1)$$

故 $\mathscr{L}\{f(t)\} = \dfrac{1}{s^2} - \dfrac{e^{-s}}{s^2}$, 對 ODE 兩端取 L–T 可得

$$s^2\hat{x}(s) - sx(0) - sx'(0) + \hat{x}(s) = \mathscr{L}\{f(t)\} = \frac{1}{s^2} - \frac{e^{-s}}{s^2}$$

其中 $\mathscr{L}\{x(t)\} = \hat{x}(s)$, 整理可得

$$\hat{x}(s) = \frac{1}{s^2(s^2+1)} - \frac{e^{-s}}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1} - \{\frac{1}{s^2} - \frac{1}{s^2+1}\}e^{-s}$$

故

$$x(t) = \mathscr{L}^{-1}\{\hat{x}(s)\} = t - \sin t - \{(t-1) - \sin(t-1)\}u(t-1)$$

4. Evaluate the line integral $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{R}$, where

$$\overrightarrow{F} = [\ln y + \cos(x)\ \cos(y)]\ \overrightarrow{i} + [\frac{x}{y} - \sin(x)\ \sin(y)]\ \overrightarrow{j}$$

is a vector field and $\overrightarrow{R} = x\,\overrightarrow{i} + y\,\overrightarrow{j}$ is the position vector in the x - y plane, for the following cases :

(a) (10%) C is a path from $(0\,,\ \frac{\pi}{2})$ to $(1\,,\ 1)$ in the domain $y > 0$.

(b) (10%) C is a simple closed path in the domain $y > 0$. 《97台大化工》

《解》

(a)

$$
\begin{aligned}
\int_C \overrightarrow{F}\cdot d\overrightarrow{R} &= \int_C [\ln y+\cos(x)\,\cos(y)]dx+[\frac{x}{y}-\sin(x)\,\sin(y)]dy \\
&= \int_C d\{x\ln y+\sin(x)\,\cos(y)\} \\
&= \{x\ln y+\sin(x)\,\cos(y)\}\Big|_{(0\,,\,\frac{\pi}{2})}^{(1\,,\,1)} \\
&= \sin 1\cdot\cos 1
\end{aligned}
$$

(b) 因 $\overrightarrow{F}$ 爲保守場, 故 $\int_C \overrightarrow{F}\cdot d\overrightarrow{R}=0$

5. Find the Fourier series representation of the following functions, both defined on $[-1\,,\,1]$:

(a) (5%) $f(x)=\begin{cases}-1 & \text{for } -1\le x<0\\ 1 & \text{for } 0\le x\le 1\end{cases}$

(b) (5%) $f(x)=\sin(5\pi x)+\cos(3\pi x)$ 《97台大化工》

《解》

(a) 因 $f(x)$ 爲奇函數, 故

$$f(x)=\sum_{n=1}^{\infty} b_n\sin(n\pi x)$$

其中

$$b_n=2\int_0^1 f(x)\sin(n\pi x)\,dx=\frac{2(1-\cos n\pi}{n\pi}$$

(b) $f(x)=\sin(5\pi x)+\cos(3\pi x)$ 即爲 $f(x)$ 在 $[-1\,,\,1]$ 中的 Fourier 級數。(不必做)

6. (20%) Solve the problem below :

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}\ ,\ -1 < x < 1\ ,\ t > 0$$

$u(-1\,,\,t) = 2,\ u(1\,,\,t) = 4,\ t > 0$

$u(x\,,\,0) = 3 + x + \sin(2\pi x)$ 《97台大化工》

《解》 令 $y = x + 1$, 故 $\begin{array}{c|c|c} x & -1 & 1 \\ \hline y & 0 & 2 \end{array}$, 則 PDE 可改寫成

$$\frac{\partial u(y,t)}{\partial t} = \frac{\partial^2 u(y,t)}{\partial y^2}\ ,\ 0 < y < 2\ ,\ t > 0 \tag{1}$$

$u(y\,,\,t)\Big|_{y=0} = 2,\ u(y\,,\,t)\Big|_{y=2} = 4$, 且

$$u(y\,,\,0) = 3 + y - 1 + \sin\{2\pi(y-1)\} = 2 + y + \sin(2\pi y)$$

再令 $u(y,t) = \phi(y,t) + y + 2$, 代入 (1) 式可得

$$\frac{\partial \phi(y,t)}{\partial t} = \frac{\partial^2 \phi(y,t)}{\partial y^2} \tag{2}$$

且由

$$u(0,t) = \phi(0,t) + 2 = 2 \ \Rightarrow\ \phi(0,t) = 0$$

$$u(2,t) = \phi(2,t) + 2 + 2 = 4 \ \Rightarrow\ \phi(2,t) = 0$$

$$u(y,0) = \phi(y,0) + y + 2 = 2 + y + \sin(2\pi y) \ \Rightarrow\ \phi(y,0) = \sin(2\pi y)$$

由 $\phi(0,t) = \phi(2,t) = 0$, 故由特徵函數展開法, 可知令

$$\phi(y,t) = \sum_{n=1}^{\infty} a_n(t) \sin\frac{n\pi y}{2}$$

代回 (2) 式可得

$$\sum_{n=1}^{\infty} a_n'(t) \sin\frac{n\pi y}{2} = \sum_{n=1}^{\infty} a_n(t)\left(-\frac{n^2\pi^2}{2^2}\right) \sin\frac{n\pi y}{2}$$

即

$$\sum_{n=1}^{\infty}\{a_n'(t)+(\frac{n\pi}{2})^2a_n(t)\}\sin\frac{n\pi y}{2}=0$$

故

$$a_n'(t)+(\frac{n\pi}{2})^2a_n(t)=0$$

則 $a_n(t)=A_ne^{-(\frac{n\pi}{2})^2t}$, 因此

$$\phi(y,t)=\sum_{n=1}^{\infty}A_ne^{-(\frac{n\pi}{2})^2t}\sin\frac{n\pi y}{2}$$

再由

$$\phi(y,0)=\sin(2\pi y)=\sum_{n=1}^{\infty}A_n\sin\frac{n\pi y}{2}$$

故 $A_4=1$, 及 $A_n=0$ $(n\neq 4)$, 因此

$$\phi(y,t)=e^{-4\pi^2t}\sin(2\pi y)$$

即

$$u(y,t)=\phi(y,t)=e^{-4\pi^2t}\sin(2\pi y)+y+2$$

故

$$u(x,t)=e^{-4\pi^2t}\sin[2\pi(x+1)]+x+1+2=e^{-4\pi^2t}\sin(2\pi x)+x+3$$